

Electrodynamics

The laws of classical electrodynamics were discovered by, Franklin, Coulomb, Ampere, Faraday and others. The person who completed the job and perfected it all in the compact and consistent form it has today, was James clerk Maxwell. The theory is now more than hundred years old. (watch <https://www.youtube.com/watch?v=AWI70HXrbG0>)

In the beginning, electricity and magnetism were entirely separate subjects. But in 1820 Oersted observed that an electric current could deflect a magnetic needle. Soon afterward, Ampere correctly postulated that all magnetic phenomena are due to electric charges in motion. Then in 1831, Faraday discovered that a moving magnet generates an electric current. By the time Maxwell and Lorentz put finishing touches on the theory, electricity and magnetism were inextricably entangled. They could no longer be regarded as separate subjects, but rather as two aspects of a single subject : electromagnetism.

(watch <https://www.youtube.com/watch?v=3HyORmBip-w>)

Faraday had speculated that light, too, is electrical in nature. Maxwell's theory provided spectacular justification for this hypothesis. Hertz presented the decisive

experimental confirmation for Maxwell's theory in 1888. By 1900, then, three great branches of physics, electricity, magnetism and optics, had merged into a single unified theory. According to the concept of the field formulation of electrodynamics, the space around an electric charge is permeated by electric and magnetic fields. A second charge, in the presence of these fields, experiences a force. The fields transmit the influence from one charge to the other. The fields mediate the interaction. (Watch *Electromagnetic Waves - with Sir Lawrence Bragg* <https://www.youtube.com/watch?v=Vwjcn4VL2iw>)

When a charge undergoes acceleration, a portion of the field “detaches” itself, in a sense, and travels off at the speed of light, carrying with it energy, momentum and angular momentum. We call this electromagnetic radiation.

Electric charge

1. Charge appears in nature in two varieties, which are called ‘plus’ and ‘minus’. Their effects (the positive and negative) tend to cancel. If we have $+q$ and $-q$ charges at the same point, electrically it is the same as having no charge there at all.
2. Charge is conserved. It cannot be created or destroyed. A plus charge can ‘annihilate’ an equal minus charge, but a plus charge cannot simply disappear

by itself. So the total charge of the universe is fixed for all time. This is called the global conservation of charge. There is a local conservation of charge as well. According to this a charge cannot disappear from one point in space and reappear at another point without a continuous path connecting the two points. The movement of the charges is defined by the continuity equation.

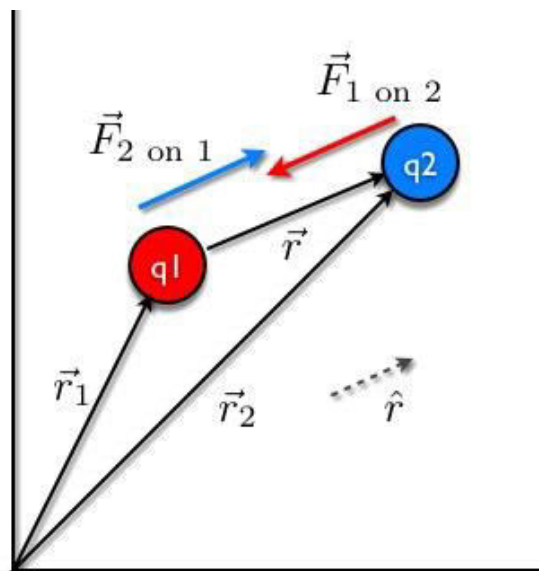
- 3 **Charge is quantized.** The electric charge appears only in discrete lumps (not necessarily required by the classical electrodynamics) as integral multiples of the basic unit of charge. If we call the charge on proton $+e$, then the charge on electron is $-e$, the neutron charge is zero, the pi mesons $+e$, 0 , and $-e$, the carbon nucleus has $+6e$ and so on. This fundamental unit of charge e is extremely small, so for practical purposes it is usually appropriate to ignore quantization altogether.

Coulomb's law

Based on experiments experiments Coulomb found that the force between two electric charges (in the form of point charges) q_1 and q_2 is given by (charges in free space)

$$F \propto \frac{q_1 q_2}{r^2} \text{ or } F = A \frac{q_1 q_2}{r^2} \dots\dots\dots(1)$$

Where r is the distance between the two point charges. A is the constant of proportionality. In c.g.s. units the unit of electric charge is defined using equation (1) Unit of electric charge is defined in c.g.s unit as that charge which, when placed at a distance of one unit (1 cm) from an identical charge repels it with a force of one unit (1 dyne). This unit in c.g.s system is called stat coulomb or 1 c.g.s unit. Under this definition A , the constant of proportionality becomes unity. But, in S.I units the unit of electric charge is already defined as coulomb. Since the unit of force, electric charge and distance are all defined independently, the constant A cannot be made to become unity. The constant $A = \frac{1}{4\pi \epsilon_0}$ in S.I. units. ϵ_0 carries dimension to balance the equation and is called the absolute permittivity of free space. The quantity 4π is intentionally included, so that all the formulae derived from equation (1) reflect the symmetry of the problem involved, and the process is called rationalization.



The direction of the force in equation (1) is along the line joining the two charges (the force is a central force). Let \vec{r}_1 be the position vector of q_1 and \vec{r}_2 the direction of the force experience by q_2 due to q_1 is

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r} \quad \text{where } r = r_2 - r_1$$

Naturally the force felt by q_1 is in the opposite direction and having same magnitude. \hat{r} is the unit vector in the direction of \vec{r} .

(For details and simulations visit

<http://web.mit.edu/viz/EM/visualizations/coursenotes/modules/guide02.pdf>)

If there is a collection of charges, q_1, q_2 etc. the force experienced by a test charge Q may be taken as the linear superposition of the forces experienced due to the individual charges. (the sum here is a vector sum)

$$\text{i.e. } F = F_1 + F_2 + \dots = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 Q \hat{r}_1}{r_1^2} + \frac{q_2 Q \hat{r}_2}{r_2^2} + \dots \right)$$

$$\text{i.e. } F = \frac{Q}{4\pi\epsilon_0} \left[\frac{q_1}{r_1^2} \hat{r}_1 + \frac{q_2}{r_2^2} \hat{r}_2 + \dots \right] \quad \dots\dots\dots(2)$$

The quantity in brackets may be considered as the combined influence of all the charges, in the configuration, on Q placed at a particular point in space. This influence of the charge configuration may be different at different points in the space around the charge configuration. This influence set up in the space around an electric charge is called an electric field. If the charge is stationary with respect to the frame of reference, the field is an electrostatic field. Electric field is represented by \vec{E} .

Thus in the above example,

$$\vec{F} = Q\vec{E} \quad \text{where } \vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{r}_i$$

\vec{E} is a function of \vec{r} the position vector of the point where \vec{E} is defined.

$$\therefore E(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{r}_i$$

Electric field is a vector quantity that varies from point to point and is determined by the configuration of source charges.

Physically $\vec{E}(\vec{r})$ is the force experienced by a unit positive charge placed at that point.

Electric field due to continuous charge distributions

Three types of continuous charge distributions may be considered.

1. Line charge distribution having charge per unit length ' λ '
2. Surface charge distribution having surface charge density ' σ '
3. Volume charge distribution having volume charge density ' ρ '

For a continuous charge distribution, the general form for the electric field must be,

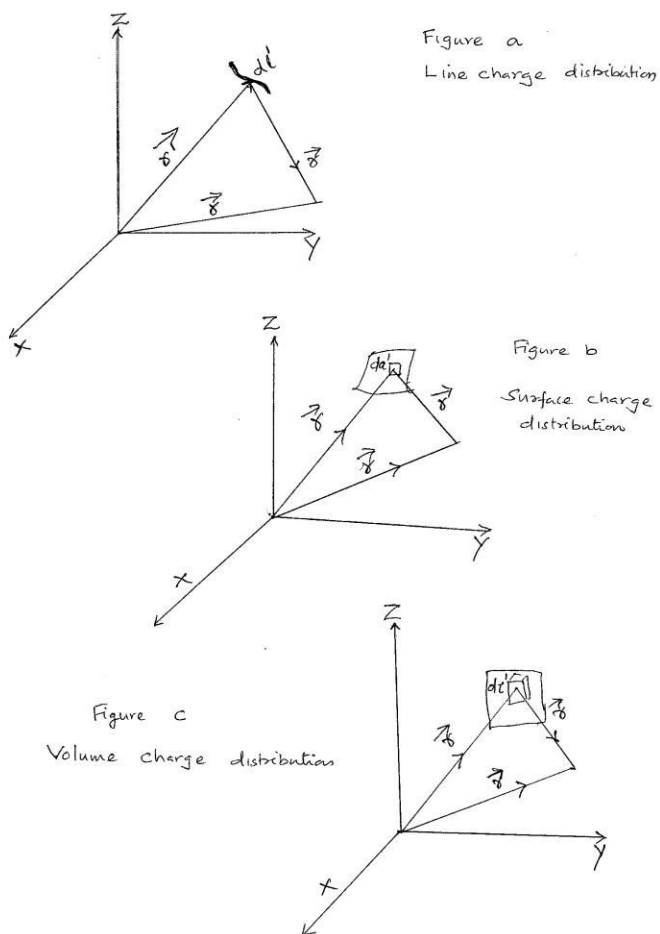
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r^2} \hat{r} dq$$

where dq is an elemental charge in the continuous distribution located at \vec{r}' and $\vec{r} = \vec{r} - \vec{r}'$. Depending upon the nature of the charge distribution dq may be expressed in terms of the charge density at \vec{r}' .

For a line charge distribution

$$dq = \lambda(\vec{r}') dl'$$

$$\therefore E(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r}') \hat{r}}{r^2} dl'$$



For a surface charge distribution (σ charge per unit area)

$dq = \sigma(\vec{r}') da'$ where da' is the elemental area

$$E(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_s \frac{\sigma(\vec{r}') \hat{r}}{r^2} da'$$

For a volume charge distribution (ρ charge per unit volume)

$$E(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho(\vec{r}') \hat{r}}{r^2} dc'$$

The calculation of electric field directly from the charge (source) distribution is based on Coulomb's law. Therefore the expressions above are sometimes called

Coulomb's law. The direct calculation may prove to be difficult often because of the complex nature of the integrals involved. Wherever symmetry of the distribution of electric charge permits, there are other easy methods for the calculation of electric field vector.

Gauss's law and the divergence of \vec{E}

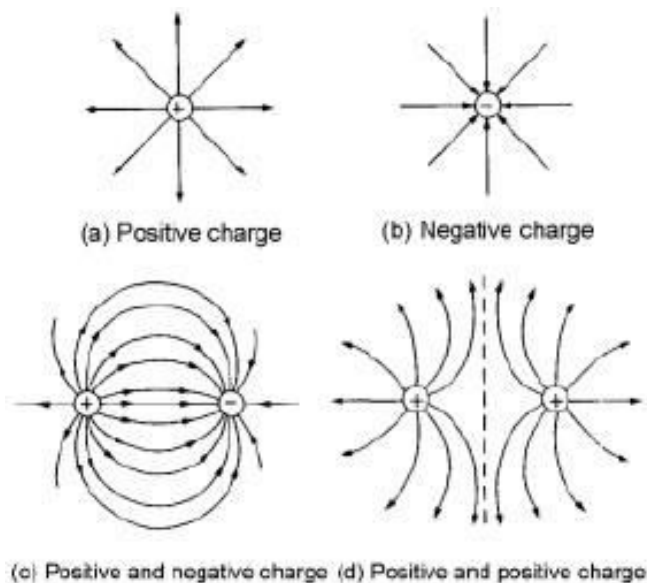
Consider the case of a point charge. The electric field around this point charge has a spherical symmetry in the sense that, in spherical polar coordinates, the strength of the electric fields depends only on the radial distance r and is independent of θ and ϕ . Therefore the magnitude of the electric field decreases as

$\frac{1}{r^2}$ from the point charge around it because,

$$E(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

In order to represent the electric field, the concept of **field lines** may be used. One can think of a few lines of force originating from the point charge and uniformly distributed around it (with spherical symmetry) with directions radially outward for a positive charge. The total number of lines should be proportional to the magnitude of the point charge. The magnitude of the electric field is given by the number of lines passing through unit area held perpendicular to the lines of force. This density of lines decreases as $\frac{1}{r^2}$; but the area increases with r^2 and the

number lines (total) remains the same. This is in conformity with the inverse square dependence of $|\vec{E}|$ with r . It should be understood that there is no stipulation regarding the total number of field lines representing a particular charge, but once a number is chosen in the scheme, then the proportionality must be maintained everywhere. For example, let N be the number of lines for a point charge q then there must be $2N$ number for a charge $2q$. The idea of electric field lines helps us in representing the electric fields due to different types of charge distributions. Field lines come out of positive charge and end in negative charge. If there only a positive charge present the field lines come out of the point charge and reach up to infinity. If there is another charge, a minus charge, present in the vicinity, some of the field lines, depending upon the magnitude of the minus charge, are directed towards it, in all cases keeping the condition that the total number of lines starting from (a + charge) or ending in (a – charge) will be proportional to the magnitude of the charge. The field lines can never cross because, if it is possible, at the point of crossing the electric field will have more than one direction!



(For simulation watch <https://www.youtube.com/watch?v=nhGqF03plsM>)

The totality of the field lines may be called the **flux** of the electric field (which naturally must be a scalar quantity, without any specification of direction). The electric flux that passes through any area of unit magnitude held perpendicular to the flux at that point is called the flux density (flux per unit perpendicular area). The flux density is the electric field \vec{E} in magnitude and direction at the point about which the perpendicular, unit area is considered. In this model the flux of \vec{E} through any surface S is

$$\phi = \int_s \vec{E} \cdot d\vec{a}$$

where \vec{E} is the field at any point on the surface and $d\vec{A}$ is the elemental area around it. The flux is a measure of the “number of lines” passing through the area S .

Visualize flux in a closed box <https://www.youtube.com/watch?v=5EN14vn82bc>

Gauss’s law states that the flux through any closed surface is $\frac{q}{\epsilon_0}$ where q is the charge enclosed by the closed surface. If there is no charge within the volume enclosed by the closed surface, the flux through the closed surface is zero. If there is net positive charge enclosed, the flux is positive, which means there is net outward ‘flow’ of flux through the closed surface. If there is net negative charge enclosed, the flux is negative, ie there is net inward ‘flow’ of flux through the closed surface.

This can be proved, considering the example of a point charge. For a point charge q , the electric field at any point is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

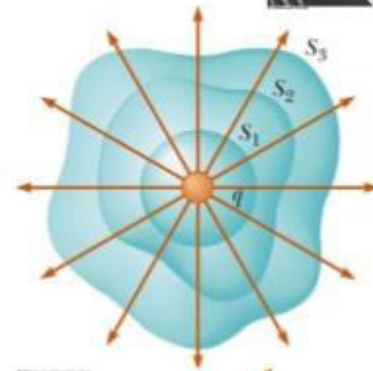
Where r is the radial distance and \hat{r} is the unit vector pointing radially outwards.

The Gaussian Surface and Gauss's Law



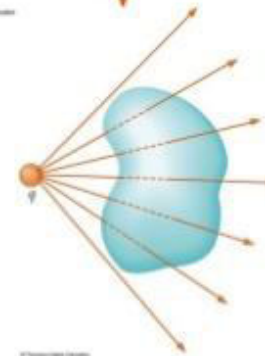
- Closed surfaces of various shapes can surround the charge
 - Only S_1 is spherical
 - The flux through all other surfaces (S_2 and S_3) are the same.
 - These surfaces are all called the Gaussian Surface.
- Gauss's Law (Karl Friedrich Gauss, 1777 – 1855):
 - The net flux through any closed surface surrounding a charge q is given by q/ϵ_0 and is independent of the shape of that surface

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$



- The net electric flux through a closed surface that surrounds no charge is zero
- Since the electric field due to many charges is the vector sum of the electric fields produced by the individual charges, the flux through any closed surface can be expressed as

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint (\vec{E}_1 + \vec{E}_2 + \dots) \cdot d\vec{A} = \frac{q_1 + q_2 + \dots}{\epsilon_0}$$



- Gauss's Law connects electric field with its source charge

Considering an elemental area, with \hat{r} perpendicular to it,

$$d\vec{A} = r^2 \sin\theta d\theta d\phi \hat{r}$$

$$\therefore \phi = \int \vec{E} \cdot d\vec{A} = \frac{1}{4\pi \epsilon_0} \int \frac{q}{r^2} \hat{r} \cdot r^2 \sin\theta d\theta d\phi \hat{r}$$

$$= \frac{q}{4\pi \epsilon_0} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{q}{4\pi \epsilon_0} \times 2 \times 2\pi$$

$$\text{i.e. } \phi = \frac{q}{\epsilon_0}$$

which is the flux through the surface of a sphere with the centre coinciding with the point charge.

In this case, the term r^2 cancels out while taking the product of area with electric field, because area increases as r^2 and \vec{E} decreases as $\frac{1}{r^2}$, always making the product a constant ($\frac{q}{\epsilon_0}$). Therefore Gauss's law is a consequence of the inverse square law nature of \vec{E} . Instead of a single charge, if there are a number of point charges (discrete), according to the principle of superposition, the total electric field is the vector sum of all the individual fields.

$$\vec{E} = \sum_{i=1}^n \vec{E}_i$$

The flux through a closed surface that encloses all the charges is,

$$\oint_s \vec{E} \cdot d\vec{a} = \sum_{i=1}^n (\oint_s \vec{E}_i \cdot d\vec{a}) = \sum_{i=1}^n \left(\frac{q_i}{\epsilon_0} \right)$$

Of there is a continuous distribution of electric charge within the volume enclosed by the closed 'Gaussian Surface', then, the Gauss's law may be stated as,

$$\oint_s \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int_v \rho d\tau$$

Where $d\tau$ is an elemental volume and ρ is the charge density in it.

Applying Gauss's theorem in vector analysis,

$$\oint_s \vec{E} \cdot d\vec{a} = \int_v (\nabla \cdot \vec{E}) d\tau$$

$$\therefore \int_V (\nabla \cdot \vec{E}) d\tau = \frac{1}{\epsilon_0} \int_V \rho d\tau$$

Since the equation is true for any arbitrary volume,

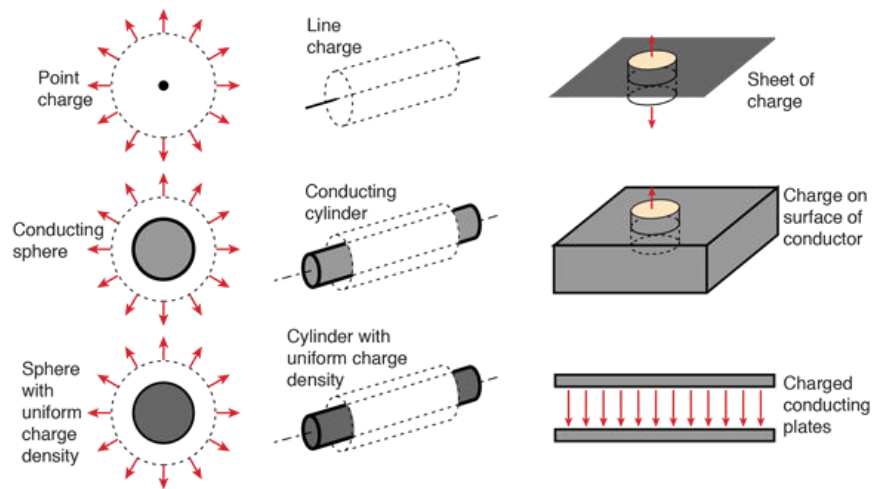
$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

This is the differential form of Gauss's law in electrostatics.

Applications of Gauss's law

Gauss's law may conveniently be used for the calculation of electric field, wherever symmetry allows. Symmetry is crucial for the easy calculation because in cases of appropriate symmetry \vec{E} can be taken out of the integral (\vec{E} being uniform in magnitude over the area of integration). In problems where such symmetry exists, Gauss's law is convenient over Coulomb's law in direct application. There are three kinds of symmetry that commonly appear in physical situations.

1. Spherical Symmetry – the Gaussian surface is a concentric sphere.
2. Cylindrical symmetry – the Gaussian surface is a coaxial cylinder.
3. Plane Symmetry – the Gaussian surface is a 'pill box' which protrudes on either side of the plane symmetrically.



The Curl of electrostatic field \vec{E}

Consider a point charge q . The radial electric field due to this charge is,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

The line integral of this field from some point a to some other point b is,

$$\int_a^b \vec{E} \cdot d\vec{l}$$

In spherical polar coordinates, the length element,

$$d\vec{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi}$$

$$\therefore \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr$$

$$\therefore \int_a^b \vec{E} \cdot d\vec{l} = \frac{q}{4\pi\epsilon_0} \int_a^b \frac{1}{r^2} dr = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r_a} - \frac{1}{r_b} \right]$$

The line integral of \vec{E} depends only on the initial and final points (ie r_a and r_b , their respective distances from the point charge situated at the origin of coordinates) Thus the line integral is independent of path. Hence the line integral of \vec{E} over a closed path is zero.

$$\text{i.e. } \oint \vec{E} \cdot d\vec{l} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = \int_s (\nabla \times \vec{E}) \cdot d\vec{a} = 0$$

Applying Stoke's theorem in vector analysis.

Since the surface is arbitrary,

$\nabla \times \vec{E} = 0$ i.e. the curl of electrostatic field \vec{E} is zero. This result is applicable to any static charge distribution.

Electric Potential

The electrostatic field \vec{E} is a very special kind of vector function, one whose curl is always zero. This special property of electrostatic field \vec{E} may be exploited to reduce a vector problem to a much simpler scalar problem. Because $\nabla \times \vec{E} = 0$, the line integral of \vec{E} around any closed loop is zero (that follows from Stoke's theorem). Because $\oint \vec{E} \cdot d\vec{l} = 0$ the line integral of \vec{E} from point a to b is the same for all paths. Because the line integral is independent of path, we can define a function,

$$V(P) = -\int_0^r \vec{E} \cdot d\vec{l}$$

where O is some standard reference point. V then depends only on the point P. V is called the electric potential. In most cases the ∞ is chosen as the reference point. The negative sign is included in the definition in order to make the potential at a point due to the presence of a positive charge as positive.

The potential difference between two points a and b is given by

$$V(b) - V(a) = -\int_{\infty}^b \vec{E} \cdot d\vec{l} - \left\{ -\int_{\infty}^a \vec{E} \cdot d\vec{l} \right\}$$

$$= -\int_{\infty}^b \vec{E} \cdot d\vec{l} - \int_{\infty}^a \vec{E} \cdot d\vec{l}$$

$$\text{i.e. } V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l}$$

Since V is a function of space coordinates, the differential,

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= \left(\frac{\partial V}{\partial x} \hat{x} + \frac{\partial V}{\partial y} \hat{y} + \frac{\partial V}{\partial z} \hat{z} \right) \cdot (dx \hat{x} + dy \hat{y} + dz \hat{z})$$

$$= \nabla V \cdot d\vec{l}$$

$$\text{But } V(b) - V(a) = \int_a^b dV = \int_a^b \nabla V \cdot d\vec{l} \quad \dots\dots\dots(2)$$

Comparing equation (1) and (2) $\vec{E} = -\nabla V$

Which means, the electrostatic field \vec{E} is the negative gradient of a scalar potential function V . According to vector analysis, the curl of a gradient is zero, which is true for an electrostatic field (ie $\nabla \times E = 0$) $\nabla \times (\nabla V) = 0$

Equipotential : A Surface over which the potential is constant is called an equipotential. The electric field cannot have, therefore, a component parallel or tangential to an equipotential surface. The electric field is always perpendicular to an equipotential surface.

Advantage of potential formulation :

If the potential V at any point is known, the field \vec{E} may be easily calculated as $\vec{E} = -\nabla V$. \vec{E} is a vector quantity having three components in a three dimensional space. But V is a scalar quantity having only one component. It is surprising to note that the three components of \vec{E} are calculated from the single component of V ! This is possible only because the curl of an electrostatic field is zero, which establishes a relation among the three components E_x , E_y and E_z .

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial z} = \frac{\partial E_z}{\partial y}$$

The potential formulation reduces a vector problem down to a scalar one.

Potential and potential difference

In the definition of potential, the choice of reference point is arbitrary. But, whatever be the choice of reference point, the potential difference remains the same, between any two points. Thus the potential may have different values at the same point depending upon the reference point. For the same reference point the potential difference remains the same and is independent of the choice of reference point. Potential as such carries no physical significance.

A point at infinity is conveniently chosen as the reference point. But in the cases of charge distribution extending upto infinity, some other reference point must be chosen.

Potential obeys superposition principle :

If V_1, V_2 it are the potentials at a point due to different charges (source) the total potential $V = V_1 + V_2 + \dots$ (an algebraic addition since V is scalar)

Poisson's equation

The differential form of Gauss's law is

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \text{ where } \rho \text{ is the volume charge density.}$$

At the same time, the electric field due to a static charge may be expressed as,
 $\vec{E} = -\nabla V$ where V is the scalar, electric potential.

Substituting \vec{E} is the Gauss's law equation.

$$\nabla \cdot (-\nabla V) = -(\nabla \cdot \nabla)V = -\nabla^2 V$$

$$\text{i.e. } -\nabla^2 V = \frac{\rho}{\epsilon_0} \quad \text{or} \quad \nabla^2 V = -\frac{\rho}{\epsilon_0}$$

This equation is called the Poisson's equation. The Poisson's equation connects the electric potential to the source charge density of potential is known, the charge density ρ may be calculated with ease, since there is only differentiation involved. But calculation of V knowing ρ is a more difficult task, if Poisson's equation is employed. The difficulty in the calculation is due to the probable complexity in the integration involved.

Laplace's equation

The Poisson equation is $\nabla^2 V = \frac{-\rho}{\epsilon_0}$. In regions where there is no charge $\rho = 0$, and

the Poisson's equation reduces to Laplace's equation.

The Laplace's equation is $\nabla^2 V = 0$

Potential of a localized charge distribution

The field \vec{E} due to a point charge is given by $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$. Then the potential at

any point in space, distant r from the origin, with reference to infinity is

$$V = -\int_{\infty}^r \vec{E} \cdot d\vec{l} = -\frac{q}{4\pi \epsilon_0} \int_{\infty}^r \frac{\pi}{r^2} dr$$

$$\text{i.e. } V = \frac{1}{4\pi \epsilon_0} \frac{q}{r}$$

Here the sign of V is positive, suggesting that the potential due to a positive charge is always positive. This has been achieved by introducing the negative sign in the definition of potential of there are a number of discrete charge q_1, q_2 etc, using the superposition principle, the potential V is,

$$V = \frac{1}{4\pi \epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i}$$

Where r_i is the distance of q_i from the point where the potential V is calculated.

Now, for a continuous charge distribution

$$V = \frac{1}{4\pi \epsilon_0} \int \frac{1}{r} dq \quad \text{where } r \text{ is the distance of } dq \text{ from the point}$$

For a volume charge distribution

$$V(\vec{r}) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(\vec{r}^1)}{r} d\tau^1$$

For a line charge

$$V(\vec{r}) = \frac{1}{4\pi \epsilon_0} \int \frac{\lambda(\vec{r}^1)}{r} dl^1$$

and for a surface charge $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r}')}{r} da'$

In the typical electrostatic problem you are given a source charge distribution ρ , and you want to find the electric field \vec{E} it produces. Unless the symmetry of the problem admits a solution by Gauss's law, it is generally advantageous to calculate the potential first as an intermediate step.

The three fundamental quantities of electrostatics are ρ , \vec{E} and V . There are six formulae interrelating them, which are actually derived from just two experimental observations (1) the principle of superposition and (2) coulomb's law.

1. From ρ to \vec{E} and V : $\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')\hat{r}}{r^2} d\tau'$

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')d\tau'}{r}$$

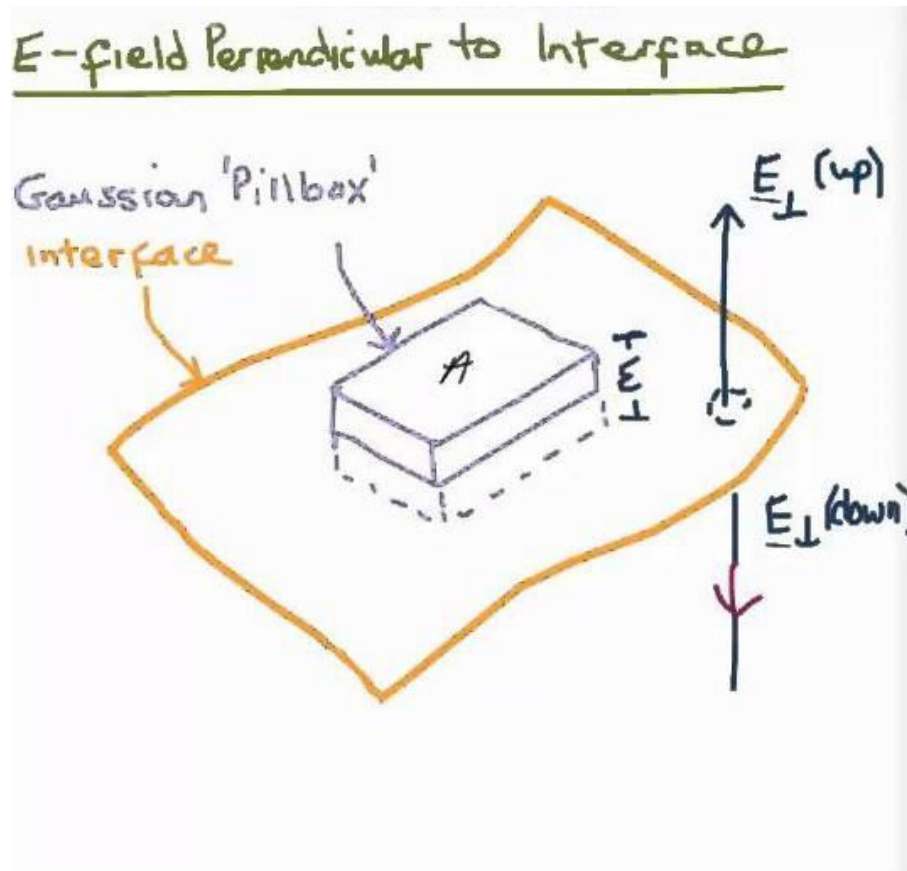
2. From \vec{E} to ρ and V : $\frac{\rho}{\epsilon_0} = \nabla \cdot \vec{E}$ $V = -\int \vec{E} \cdot d\vec{l}$

3. From V to ρ and \vec{E} : $\frac{-\rho}{\epsilon_0} = \nabla^2 V$ $\vec{E} = -\nabla V$

Electrostatic Boundary Conditions

Electric field always undergoes a discontinuity when it crosses a surface of charge density σ . Consider a thin sheet carrying a surface charge density σ . Let us draw a

wafer thin Gaussian pill box, extending just barely over the edge in each direction
(above and below)



According to Gauss's law

$$\oint_s \vec{E} \cdot d\vec{a} = \frac{Q_{\text{ena}}}{\epsilon_0} = \frac{1}{\epsilon_0} \sigma A$$

where A is the area of the pill box lid. (of σ is not uniform over the surface, we must consider A to be very small).

Since the sides of the pill box have extremely small area, they do not contribute to the flux. Therefore, the equation is

$$E_{\text{above}}^{\perp} A - E_{\text{below}}^{\perp} A = \frac{\sigma A}{\epsilon_0}$$

$$\text{ie. } E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp} = \frac{\sigma}{\epsilon_0}$$

Where E_{above}^{\perp} denotes the component of \vec{E} that is perpendicular to the surface immediately above and E_{below}^{\perp} is the component perpendicular to the surface just below. For consistency the upward direction is considered as positive in both the cases. Now, the conclusion is that, the normal component of the electrostatic field is discontinuous by an amount $\frac{\sigma}{\epsilon_0}$ at any boundary. However, where there is no surface charge, E^{\perp} is continuous.

The tangential component of \vec{E} , by contrast, is always continuous. To prove this consider

$$\oint \vec{E} \cdot d\vec{l} = 0$$

Considering dl as the length element along the closed loop on the side of the pill box,

$$\oint \vec{E} \cdot d\vec{l} = E_{\text{above}}^{\parallel} l - E_{\text{below}}^{\parallel} l = 0$$

Since, the thickness of the pill box $\epsilon \rightarrow 0$

$\therefore E_{\text{above}}^{11} = E_{\text{below}}^{11}$ where E^{11} stands for the components parallel (ie tangential) to the surface. The boundary conditions on \vec{E} can be combined into a single formula

$$\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{n}$$

Where \hat{n} is a unit vector perpendicular to the surface pointing from “below” to “above”.

The potential is continuous across any boundary since

$$V_{\text{above}} - V_{\text{below}} = -a \int_a^b \vec{E} \cdot d\vec{l}$$

As $\epsilon \rightarrow 0$ the path length shrinks to zero so that the integral tends to zero.

$$\therefore V_{\text{above}} = V_{\text{below}}$$

The gradient of the potential V is discontinuous as \vec{E} is discontinuous.

$$\text{Ie. } \nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0} \sigma \hat{n}$$

Work and Energy in electrostatics

1. The work done to move a charge : Consider a charge Q in an electric field \vec{E} . The charge experiences a force $Q\vec{E}$ in the direction of the field. Of now the charge

is to be moved in a direction opposite to the field, without acquiring any acceleration, a force $\vec{F} = -Q\vec{E}$ is to be applied and if the charge moves through a distance (infinitesimal displacement) $d\vec{l}$ under this force the work done on the charge is $dw = \vec{F} \cdot d\vec{l} = -Q\vec{E} \cdot d\vec{l}$. For a finite displacement from point a to point b, the work done is,

$$W = -Q \int_a^b \vec{E} \cdot d\vec{l} \quad \text{But } - \int_a^b \vec{E} \cdot d\vec{l} = V(b) - V(a)$$

Where $V(b)$ and $V(a)$ are the electrostatic potentials at b and a respectively.

This work gets stored as potential energy of the charged body in the electrostatic field. $QV(b)$ may be called the potential energy of the charge at the point 'b' and $QV(a)$ the potential energy at 'a'. Under this concept, the electrostatic potential $V(b)$ or $V(a)$ may be interpreted as 'potential energy' per unit charge. And, potential difference is understood as the workdone in moving unit charge from one point to the other.

$$V(b) - V(a) = \frac{W}{Q}$$

Now, if the zero potential is considered at infinity, the workdone in bringing the charge from infinity to a point P is given by

$$W = Q[V(P) - V(\infty)]$$

ie $W = QV(r)$ Since $V(\infty) = 0$ according to the choice of reference point at infinity.

2. The energy of a point charge distribution

Consider a certain configuration of point charges. We can imagine that all these charges have been brought from infinity (say, the zero, potential) to their respective positions. Thus work is done to assemble all the point charges into the configuration.

To start with, let charge q_1 be brought from infinity to the position r_1 . Since there is no electric field (in the absence of any charge) already existing, no work is done in this process. Therefore, the work $W_1 = 0$. Now, we bring charge q_2 and place it at position r_2 . Since q_1 is already there, there is a potential at r_2 given by,

$$V(r_2) \text{ due to } q_1 \text{ at } r_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{12}}$$

Where $r_{12} = r_2 - r_1$. Therefore work done,

$$W_2 = q_2 \times \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{12}} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}$$

In bringing the charge q_3 from infinity and placing it at r_3 is

$$W_3 = \frac{1}{4\pi\epsilon_0} \left[\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right]$$

Similarly for a charge q_4 at r_4

$$W_4 = \frac{1}{4\pi \epsilon_0} \left[\frac{q_1 q_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}} \right]$$

Thus the total work done in assembling all the four charges is,

$$\begin{aligned} W &= W_1 + W_2 + W_3 + W_4 \\ &= 0 + \frac{1}{4\pi \epsilon_0} \frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi \epsilon_0} \left[\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right] + \frac{1}{4\pi \epsilon_0} \left[\frac{q_1 q_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}} \right] \\ \text{ie } W &= \frac{1}{4\pi \epsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}} \right] \end{aligned}$$

If there are n charges in the configuration, the work done in assembling them is,

$$W = \frac{1}{4\pi \epsilon_0} \sum_{\substack{j=1 \\ j>i}}^n \frac{q_i q_j}{r_{ij}}$$

The stipulation $j > i$ is to ensure that doubling of the terms does

not occur.

Another way to do this is to count each pair twice and take half of the sum.

$$\text{ie } W = \frac{1}{2} \frac{1}{4\pi \epsilon_0} \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{q_i q_j}{r_{ij}}$$

In this representation since all the combinations come, the order in which the assembly is made is insignificant

Now, making some rearrangements,

$$W = \frac{1}{2} \sum_{i=1}^n q_i \left(\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{4\pi \epsilon_0} \frac{q_j}{r_{ij}} \right)$$

The term in parantheses is the potential at r_i (the position of q_i) due to all other

charges. If this potential is called $V(r_i)$, $W = \frac{1}{2} \sum_{i=1}^n q_i V(r_i)$

This is the energy possessed by the configuration of charges.

3. The energy of a continuous charge distribution

If ρ is the volume charge density for a three dimensional space, the energy possessed by it as the work done in assembling the continuous charge distribution, is given by

$$W = \frac{1}{2} \int \rho V d\tau$$

Where in general, ρ and V may be functions of position coordinates.

From Gauss's law

$$\rho = \epsilon_0 \nabla \cdot \vec{E}$$

$$\therefore W = \frac{\epsilon_0}{2} \int (\nabla \cdot \vec{E}) V d\tau$$

We have

$$\nabla \cdot (V\vec{E}) = V\nabla \cdot \vec{E} + \vec{E} \cdot \nabla V$$

$$\therefore \int \nabla \cdot (V\vec{E}) d\tau = \int V(\nabla \cdot \vec{E}) d\tau + \int \vec{E} \cdot (\nabla V) d\tau$$

but according to Gauss's divergence theorem,

$$\int_V \nabla \cdot (V\vec{E}) d\tau = \int_S V\vec{E} \cdot d\vec{a}$$

$$\therefore \oint_s \vec{V}\vec{E}.d\vec{a}^0 = \int V(\nabla.\vec{E})d\tau + \int \vec{E}.(\nabla V)d\tau$$

$$\therefore \int (\nabla.\vec{E})Vd\tau = -\int \vec{E}.(\nabla V)d\tau + \oint_s \vec{V}\vec{E}.d\vec{a}^0$$

$$\therefore W = \frac{\epsilon_0}{2} \left[-\int \vec{E}.(\nabla V)d\tau + \oint_s \vec{V}\vec{E}.d\vec{a}^0 \right]$$

$$\text{But } -\nabla V = \vec{E}$$

$$\therefore W = \frac{\epsilon_0}{2} \left[\int E^2 d\tau + \oint_s \vec{V}\vec{E}.d\vec{a}^0 \right]$$

Of the volume over which the integral is done is increased the value of the first integral increases (since E^2 is positive) But in order to make the sum finite the second integral decreases (since V decreases as $\frac{1}{r}$ and E decreases as $\frac{1}{r^2}$ as the surface gets enlarged, but the surface increases as r^2 making the net decrease of the integral as $\frac{1}{r}$)

Thus, making the integral over all space, the second integral becomes zero.

$$\therefore W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$

The energy stored per unit volume (energy density) of the field $= \frac{1}{2} \epsilon_0 E^2$

The electrostatic energy is quadratic in the field. \vec{E} . Therefore the energy does not obey a superposition principle.

The energy of a compound system is not the sum of the energies of its parts considered separately. (there are also cross terms)

Consider the superposition of two fields \vec{E}_1 and \vec{E}_2 .

The total field is $\vec{E} = \vec{E}_1 + \vec{E}_2$.

The total energy is $W = \frac{\epsilon_0}{2} \int E^2 d\tau = \frac{\epsilon_0}{2} \int (\vec{E} \cdot \vec{E}) d\tau$

$$\text{ie. } W = \frac{\epsilon_0}{2} \int (\vec{E}_1 + \vec{E}_2) \cdot (\vec{E}_1 + \vec{E}_2) d\tau$$

$$\begin{aligned} \text{ie. } W &= \frac{\epsilon_0}{2} \int E_1^2 d\tau + \frac{\epsilon_0}{2} \int E_2^2 d\tau + \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d\tau \\ &= W_1 + W_2 + \epsilon_0 \int (\vec{E}_1 \cdot \vec{E}_2) d\tau \end{aligned}$$

If the charge everywhere is doubled the energy is quadrupled.

Conductors

In an insulator, such as glass or rubber, each electron is attached to a particular atom. In a metallic conductor, by contrast, one or more electrons per atom are free to move about through the material. Thus a conductor has a large number of ‘free electrons’ or ‘conduction electrons’

1. The electric field $\vec{E} = 0$ inside a conductor

When a conductor is exposed to an external electric field \vec{E}_0 , the field will drive positive charges in the direction of the field and negative charges in the opposite direction. Since there are only free electrons in the conductor, the electric field \vec{E}_0 will drive the electrons in a direction opposite to the electric field making the side on the direction of the electric field positive as shown in fig. The side where the electrons are driven to becomes negative. These are called induced charges. Thus it is seen that the induced charges produce an electric field \vec{E}_1 , opposite to \vec{E}_0 , inside the conductor. The charge flow inside the conductor will continue until the induced field \vec{E}_1 becomes equal in magnitude to \vec{E}_0 . Since the two fields are equal and opposite, they cancel out each other making the electric field inside a conductor zero.

2. $\rho = 0$ inside a conductor

This follows from Gauss's law : $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ or $\vec{E} = 0$, ρ is also zero charges are

there inside a conductor, but exactly as much plus charge as minus, so the net charge density (macroscopically averaged) in the interior of a conductor is zero.

3. Any net charge resides on the surface

4. A conductor is an equipotential

If a and b are any two points within or on the surface of a conductor then,

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l} = 0 \quad \text{since } \vec{E} = 0$$

$\therefore V(b) = V(a)$ Thus the potential is the same every where inside and on the surface of a conductor.

5. \vec{E} is perpendicular to the surface, just outside a conductor

If there happens to be a tangential component of \vec{E} , this will drive charges in such a way that an induced field will develop, cancelling the tangential component of \vec{E} . There will be only a perpendicular \vec{E} just outside.

Electric field inside a dielectric

Most everyday objects belong to one of the two large classes: conductors and insulators (or dielectrics). In dielectrics, all charges are attached to specific atoms or molecules. What such charges can maximum do is to move a bit within the atom or molecule.

When an atom is placed in an electric field, the nucleus is pushed in the direction of the field, and the electrons the opposite way. The two opposing forces - \vec{E} pulling the nucleus and electrons apart, their mutual attraction drawing them together - reach a balance, leaving the atom polarized. The atom now has a tiny dipole moment \vec{P} , which points in the same direction as \vec{E} . Typically, this induced

dipole moment is approximately proportional to the field as long as the field is not too strong.

$$\text{i.e. } \vec{P} = \alpha \vec{E}$$

The constant of proportionality α is called atomic polarizability.

For molecules the situation is not quite so simple. The polarizability may be different in different directions in a molecule, making α a tensor rather than a scalar as in the above case.

If the electric field applied is too large, it may ionize the atoms in a dielectric breakdown.

Polarization

If a substance consisting of neutral atoms is placed in an electric field, polarization takes place. If the material is made up of polar molecules (having permanent dipole moment) each permanent dipole will experience a torque, tending to line it up along the field direction. These two mechanisms produce the same result a lot of little dipoles pointing along the direction of the field make the material polarized.

A convenient measure of this effect is called polarization.

Polarization \vec{P} is the dipole moment per unit volume

The electric displacement

Gauss's law in the presence of dielectrics

Within a dielectric, the charge density has two parts one is the bound charge (in the dipole) density ρ_b and the other the free charge (if present) density ρ_f . The total charge density is

$$\rho = \rho_b + \rho_f$$

It can be shown that $\rho_b = -\nabla \cdot \vec{P}$

From Gauss's law $\rho = \epsilon_0 \nabla \cdot \vec{E}$

$$\therefore \epsilon_0 \nabla \cdot \vec{E} = -\nabla \cdot \vec{P} + \rho_f$$

$$\text{i.e. } \epsilon_0 \nabla \cdot \vec{E} + \nabla \cdot \vec{P} = \rho_f$$

$$\text{i.e. } \nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

The expression $\epsilon_0 \vec{E} + \vec{P}$ is defined as the electric displacement \vec{D} . Therefore the Gauss's law reads,

$$\nabla \cdot \vec{D} = \rho_f$$

or in integral form $\oint_S \vec{D} \cdot d\vec{a} = Q_{\text{fene}}$ where Q_{fene} is the total free charge enclosed in

the volume. This is a particularly useful way to express Gauss's law because it makes reference only to free charges. Free charges are the source of electric field, bound charges come as an after effect.